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FINAL

جامعة بيرزيت
BIRZEIT UNIVERSITY

Mathematics Department

99

٩٩ Math 330

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Part I(each problem worth 7 points)

(1) Given the Taylor polynomial expansions

$$\cos(h) = 1 - \frac{h^2}{2} + \frac{h^4}{4!} + O(h^6)$$

$$\sin(h) = h - \frac{h^3}{3!} + \frac{h^5}{5!} + O(h^7)$$

Determine the order of approximation for their sum and product.

3 $\cos(h) + \sin(h) = 1 + h - \frac{h^2}{2!} - \frac{h^3}{3!} + \frac{h^4}{4!} + \frac{h^5}{5!} + o(h^6)$

their sum $o(h^6)$

$\frac{2\sin h \cos h}{2} = \frac{1}{2}\sin 2h = \frac{1}{2}\left[2h - \frac{8h^3}{3!} + \frac{32h^5}{5!} + o((2h)^7)\right]$

2 $\frac{2\sin h\cos h}{2} = \frac{1}{2}\left(2h - \frac{8h^3}{3!} + \frac{32h^5}{5!} + 2^7 o(h^7)\right)$

the order of approx. of thier product $o(h^7)$

(2) Investigate the nature of the fixed point iteration for each of the fixed points for

7

$$g(x) = -4 + 4x - \frac{x^2}{2}$$

$-\frac{x^2}{2} + 4x - 4 = x$

~~$\frac{x^2}{2} - 3x + 4 = 0$~~

~~$x^2 - 6x + 8 = 0$~~

~~$(x-2)(x-4) = 0$~~

$x = 2$

~~$g(x) = x$~~ $g'(x) = 4 - x$

$|g'(x)| < 1 \Rightarrow |4 - x| < 1$

$-1 < 4 - x < 1$

$-5 < -x < -3$

$3 < x < 5$

$f(x) = x^2 - 6x + 8 = 0$

$x = \frac{1}{8}[x^2 + 8]$

$x = g(x)$

$g_2'(x) = \frac{1}{3}$

$g_2'(2)$

7

(3) Find the order of the error of the formula

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(c)$ using the taylor expand at x

$$f'(x) = \frac{f(x+h)-f(x)}{h} - \frac{h}{2!} f''(c)$$

$\therefore$ the order $O(h)$

(4) Suppose that you save money by making regular monthly deposits P and the annual interest rate is I, then the total amount A after n deposits is given by

7

$$A = \frac{P}{I/12}[(1+I/12)^n - 1]$$

$500{,}000$ 300 240

Find an approximation for the interest rate I that will yield the total amount of \$500,000 if 240 monthly payments of \$300 are made.

Find the first two iterations using the bisection method starting with [a,b]=[0.15,0.16]

Define the function

$$f(x) = \frac{300}{x/12}\left(\left(1+\frac{x}{12}\right)^{240} - 1\right) - 500{,}000 = 0$$

$$f(x) = \frac{3600}{x}\left(\left(1+\frac{x}{12}\right)^{240} - 1\right) - 500{,}000$$

$a_0 = 0.15 \qquad b_0 = 0.16 \qquad f(0.15) = -50828.16 < 0$

$f(0.16) = 17937.5 > 0$

$$c = \frac{a_0+b_0}{2} = 0.155$$

$f(0.155) = -17793.597 < 0$

$a_1 = 0.155 \qquad b_1 = 0.16 \qquad f(0.155) < 0 \qquad f(0.16) > 0$

$c = \frac{0.155+0.16}{2} = 0.1575 \qquad f(0.1575) = -[illegible] < 0$

(5) If $x = \frac{4x - x^2 + y + 3}{4} = g_1(x,y)$

$y = \frac{3 - xy + 2y}{2} = g_2(x,y)$

7

Using Gause Seidel fixed point iteration, find (p_1, q_1)
Given $(p_0, q_0) = (2.1, 1.4)$

$$p_{i+1} = g_1(p_i, q_i) \qquad q_{i+1} = g_2(p_{i+1}, q_i)$$

$$p_1 = g_1(p_0, q_0) = g_1(2.1, 1.4) = 2.0975$$

$$q_1 = g_2(p_1, q_0) = g_2(2.0975, 1.4) = 1.43175$$

$$\therefore \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 2.0975 \\ 1.43175 \end{pmatrix}$$

(6) Given L is a lower nxn triangular matrix, where the diagonal elements are all ones.

Find the total cost of the forward substitution
Lx =b

take $n = 4$ then generalize

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Forward substitution

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(5)

(7) Let $f(x) = 2\sin\left(\frac{\pi x}{6}\right)$ where x is in radians 7

(a) Use quadratic Lagrange interpolation based on the nodes $x_0 = 0$, $x_1 = 1, x_2 = 2$ to approximate $f(1.5)$

(b) Find the best upper bound for the error in the above estimation.

$$P_2(x) = \sum_{k=0} L_{2k} f(x_k)$$

$$P_2(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$P_2(1.5) = f(x_0)\left[\frac{(1.5-1)(1.5-2)}{(0-1)(0-2)}\right] + f(x_1)\left[\frac{(1.5-0)(1.5-2)}{(1-0)(1-2)}\right] + f(x_2)\frac{(1.5-0)(1.5-1)}{(2-0)(2-1)}$$

$f(x_0) = f(0) = 2\sin 0 = 0$

$f(x_1) = f(1) = 2\sin\frac{\pi}{6} = 1$

$f(x_2) = f(2) = 2\sin\frac{\pi}{3} = \sqrt{3}$

$P_2(1.5) = (f(1) \times 0.75) + (f(2) \times 0.375) = 1.399519053$

$\therefore f(1.5) = 1.399519053$

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4

(3)

Part II(each problem worth 13 points) 13

(8) (a) Drive the normal equation of the power fit of the form
$y = a\,t^m$ where m is fixed.

$E(a) = \sum (f(t_i) - y_i)^2$

$E(a) = \sum (a t_i^m - y_i)^2$

$\frac{\partial E}{\partial a} = 2 \sum (a t_i^m - y_i) t_i^m = 2 \sum (a t_i^{2m} - y_i t_i^m) = 0$

$a \sum t_i^{2m} - \sum y_i t_i^m = 0 \Rightarrow \boxed{a = \frac{\sum y_i t_i^m}{\sum t_i^{2m}}}$

(b) Use the power fit above to estimate the gravitational constant g where $d = \frac{1}{2} g t^2$ using the following data

Time, t_k	Distance d_k
0.2	0.1960
0.4	0.7835
0.6	1.7630

$d = \frac{1}{2} g t^2 \Rightarrow y = a t^2$

$a = \frac{\sum y_i t_i^2}{\sum t_i^4}$

t_i	y_i	t_i^2	t_i^4	$y_i t_i^2$
0.2	0.1960	0.04	1.6×10^{-3}	7.84×10^{-3}
0.4	0.7835	0.16	0.0256	0.12536
0.6	1.7630	0.36	0.1296	0.63468
			0.1568	0.76788

(9)

(9) Let $A = \int_1^2 e^{-x} dx$

(1) Estimate A using Trapezoidal Rule.
(2) Estimate A using Simpson's 1/3 Rule.
(3) Estimate A using Gauss Legendre 2 points formula.
(4) Which one of the above is most accurate.

13

1) $A = \int_1^2 e^{-x} dx = \frac{h}{2}[f_0 + f_1]$

$h = \frac{b-a}{n} = 1 \quad x_0 = 1, \; x_1 = 2. \Rightarrow h = 1$

$A_T = \frac{1}{2}[f(1) + f(2)] = \frac{1}{2}[e^{-1} + e^{-2}] = 0.2516073362$

2

2) $A = \int_1^2 e^{-x} dx = \frac{h}{3}[f_0 + 4f_1 + f_2)$

$h = \frac{b-a}{n} = \frac{2-1}{2} = 0.5$

$A_{S_{1/3}} = \frac{0.5}{3}[f(1) + 4f(1.5) + f(2)] = \frac{0.5}{3}[e^{-1} + 4e^{-1.5} + e^{-2}]$

$A_{S_{1/3}} = 0.2332621256$.

2

[illegible]

(10) Consider the IVP

13

$$y' = e^{-2t} - 2y$$
$$y(0) = 0.1$$

Using $h = 0.1$

Estimate $y(0.1)$ using

1) Euler method
2) Taylor method of $O(h^2)$
3) Runge-Kutta method of $O(h^4)$
4) Which of the above is the most accurate given that the solution is

$$y(t) = \frac{1}{10}e^{-2t} + te^{-2t} \Rightarrow \text{Exact}$$

$y(0.1) = 0.16374615$

$t_k = t_0 + hk$

$t_1 = 0 + h = 0.1$

1) $y_{i+1} = y_i + h f(t_i, y_i)$

$y_1 = y_0 + h f(t_0, y_0)$

$y_1 = 0.1 + h f(0, 0.1) = 0.1 + 0.1\left[e^{-2(0)} - 2(0.1)\right]$

$y_1 = y(0.1) = 0.18$

~~Exact~~ $\epsilon_1 = |y(0.1) - y_1| = 0.01625384 9.$

2) $y_{i+1} = y_i + h d_1 + \frac{h^2}{2!} d_2.$

$y'(t) = e^{-2t} - 2y.$

$y''(t) = (e^{-2t} - 2y)_t + (e^{-2t} - 2y)_y (e^{-2t} - 2y)$

$y''(t) = -2e^{-2t} - 2(e^{-2t} - 2y) = -4e^{-2t} + 4y$

13

(11) The partial derivative $f_x(x,y)$ is obtained by holding y fixed and differentiating with respect to x. Use the formula $g'(x) \approx \frac{g(x+h)-g(x-h)}{2h}$ to find a formula to estimate $f_x(x,y)$ and $f_y(x,y)$.

b) Let $f(x,y)=\frac{xy}{x+y}$. Use the formula found in (a) to estimate the differential $df = f_x(x,y)dx + f_y(x,y)dy$ at $(2,3)$ using $dx = dy = 0.1$

a) $g'(x) = \frac{g(x+h)-g(x-h)}{2h}$

$f_x(x,y) = \frac{f(x+h,y)-f(x-h,y)}{2h}$

$f_y(x,y) = \frac{f(x,y+h)-f(x,y-h)}{2h}$

b) $f_x(2,3) = \frac{f(2+0.1,3)-f(2-0.1,3)}{2(0.1)} = \frac{f(2.1,3)-f(1.9,3)}{0.2}$

$f_x(2,3) = 0.360144057$

$f_y(2,3) = \frac{f(2,3.1)-f(2,2.9)}{0.2} = 0.160064025$

$df\Big|_{(2,3)} = f_x(2,3)\,dx + f_y(2,3)\,dy = 0.1(f_x(2,3)+f_y(2,3))$

$\boxed{df = 0.0520208808}$